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Primitive one-factorizations and the geometry of mixed translations☆

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Abstract

We construct an infinite family of one-factorizations of K_v admitting an automorphism group acting primitively on the set of vertices but no such group acting doubly transitively. We also give examples of one-factorizations which are *live*, in the sense that every one-factor induces an automorphism, but do not coincide with the affine line parallelism of $AG(n, 2)$. To this purpose we develop the notion of a "mixed translation" in $AG(n, 2)$.

MSC: 05C70; 05B25; 51D20

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1. Introduction

There is a fairly straightforward connection between one-factorizations of the complete graph K_v, v even, and certain permutation sets of degree v. This connection has been pointed out at various places, for example [2, Section 2]: we need to repeat a few things here in order to set the notation for the subsequent sections.

A warning before we begin. Some identical terms are used with different meanings for graphs and for permutations. A typical example is the term "degree". We shall avoid giving explicit definitions of such standard terms, which can be found in textbooks like [3, Section 1.2, p. 5; 5, Section 1.3, p. 6].

It is clear that a one-factor of K_v, v even, determines a fixed-point-free involution on the vertices of K_v and conversely: when we write the edges occurring in the one-factor as 2-cycles, then the involution is precisely the product of the 2-cycles arising from the edges of the one-factor.

If a one-factorization $\mathscr{F}$ of K_v is given, then the fixed-point-free involutions arising from the one-factors together with the identity form a sharply transitive permutation set of degree v. If, conversely, a sharply transitive permutation set of even degree v is given, which consists of the identity and of $v-1$ fixed-point-free involutions, then a one-factorization of K_v is obtained.

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If $\mathscr{F}$ is the one-factorization and Φ is the corresponding set of fixed-point-free involutions together with the identity then the automorphism group of $\mathscr{F}$ coincides with the normalizer $N_{\Sigma}(\Phi)$ of Φ in the full symmetric group Σ on $V = V(K_v)$, see [2, p. 2], and it also coincides with $N_{\Sigma}(\Phi \backslash \{\mathrm{id}_V\})$.

We shall thus generally not distinguish between the one-factorization and the corresponding set of fixed-point-free involutions. Indeed, with a little abuse of notation, we shall only speak of $\mathscr{F}$ and the elements of $\mathscr{F}$ will be treated as one-factors or fixed-point-free involutions according to our convenience.

A one-factorization $\mathscr{F}$ of K_v, v even, is said to be *primitive* if there exists an automorphism group G of $\mathscr{F}$ acting primitively on the vertices of K_v. If G acts doubly transitively on the vertices of K_v then we say that $\mathscr{F}$ is a *doubly transitive* one-factorization. The doubly transitive one-factorizations of complete graphs are classified by the work of Cameron and Korchmàros [2, Theorem 3]. If $\mathscr{F}$ is such a doubly transitive one-factorization then v is a power of 2, with the only exceptions of $v = 6$, 12 and 28. Furthermore, for the non-exceptional values of v, the one-factorization $\mathscr{F}$ can always be described as arising from the affine line parallelism of an affine space over the field of two elements.

Do there exist primitive one-factorizations of K_v which are not doubly transitive? We shall see in Section 3 that the answer to this question is affirmative. Is it possible to construct examples for such one-factorizations for infinitely many values of v? We shall see that the answer to this question is also affirmative.

If $\mathscr{F}$ is a given one-factorization of K_v and g is a one-factor in $\mathscr{F}$ it may well happen that, when we regard g as a permutation, it is an automorphism of $\mathscr{F}$. If that is the case for *all* one-factors in $\mathscr{F}$ then we say that $\mathscr{F}$ is a *live* one-factorization. This terminology was used by Cameron in a seminar during the Summer School on Finite Geometries held at the Università della Basilicata, Potenza, Italy, in 1999. Again, the one-factorizations arising from the affine line parallelism of $AG(n, 2)$ are examples of *live* one-factorizations.

Are there other examples? The answer is yes and we shall give some in this paper. Our constructions are geometric in nature. We work namely in the affine space $AG(n, 2)$ and develop the notion of a “mixed translation”: roughly speaking this is a transformation acting on half of the points as a translation in one direction and on the other half as a translation in another direction. In characteristic 2 these transformations are fixed-point-free involutions and are thus good candidates for one-factors.

2. Mixed translations over $GF(2)$

Let $F = GF(2)$ be the field of two elements and let $V = F^n$ be the n-dimensional vector space over F. For $\mathbf{a} \in V$ let $t_{\mathbf{a}}$ denote the translation determined by the vector $\mathbf{a}$. We have thus $t_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ for $\mathbf{x} \in V$. We denote by T the translation group on V, that is, $T = \{t_{\mathbf{a}} : \mathbf{a} \in V\}$.

Let W be a hyperplane of V, that is, an $(n-1)$-dimensional vector subspace of V. We define $\overline{W} = V \backslash W$ and we have $\overline{W} = \mathbf{y} + W$ for each vector $\mathbf{y} \in V \backslash W$. Let $\mathbf{w}_1, \mathbf{w}_2$ be linearly independent vectors in W. Define a fixed-point-free involution m on V by setting

$$m(\mathbf{x}) = \begin{cases} \mathbf{x} + \mathbf{w}_1 & \text{if } \mathbf{x} \in W, \\ \mathbf{x} + \mathbf{w}_2 & \text{if } \mathbf{x} \in \overline{W}. \end{cases}$$

We call m a *mixed translation*. Clearly, m depends on the choice of the hyperplane W and of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$ within W, and we should rather write $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$. We shall omit reference to W, $\mathbf{w}_1$, $\mathbf{w}_2$ whenever possible. The *complementary* mixed translation $\overline{m} = \overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2}$ is defined by setting

$$\overline{m}(\mathbf{x}) = \begin{cases} \mathbf{x} + \mathbf{w}_1 & \text{if } \mathbf{x} \in \overline{W}, \\ \mathbf{x} + \mathbf{w}_2 & \text{if } \mathbf{x} \in W. \end{cases}$$

We shall now establish some computational rules for mixed translations which we shall use in later sections.

Proposition 1. *We have* $m_{W,\mathbf{w}_1,\mathbf{w}_2}\, \overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2} = \overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2}\, m_{W,\mathbf{w}_1,\mathbf{w}_2} = t_{\mathbf{w}_1+\mathbf{w}_2}$.

Proof. If $\mathbf{x} \in W$, then we have $\overline{m}m(\mathbf{x}) = \overline{m}(\mathbf{x} + \mathbf{w}_1) = \mathbf{x} + \mathbf{w}_1 + \mathbf{w}_2 = t_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{x}) = m(\mathbf{x} + \mathbf{w}_2) = m\overline{m}(\mathbf{x})$. If $\mathbf{x} \in \overline{W}$, then we have $\overline{m}m(\mathbf{x}) = \overline{m}(\mathbf{x} + \mathbf{w}_2) = \mathbf{x} + \mathbf{w}_1 + \mathbf{w}_2 = t_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{x}) = m(\mathbf{x} + \mathbf{w}_2) = m\overline{m}(\mathbf{x})$. $\square$

Proposition 2. *The following relations hold*:

$$t_{\mathbf{a}} m t_{\mathbf{a}} = \begin{cases} m & \text{if } \mathbf{a} \in W, \\ \overline{m} & \text{if } \mathbf{a} \in \overline{W}, \end{cases} \qquad t_{\mathbf{a}} \overline{m} t_{\mathbf{a}} = \begin{cases} \overline{m} & \text{if } \mathbf{a} \in W, \\ m & \text{if } \mathbf{a} \in \overline{W}. \end{cases}$$

Proof. We have

$$t_{\mathbf{a}} m t_{\mathbf{a}}(\mathbf{x}) = m(\mathbf{x}+\mathbf{a}) + \mathbf{a} = \begin{cases} \mathbf{x}+\mathbf{a}+\mathbf{w}_1+\mathbf{a} & \text{if } \mathbf{x}+\mathbf{a} \in W \\ \mathbf{x}+\mathbf{a}+\mathbf{w}_2+\mathbf{a} & \text{if } \mathbf{x}+\mathbf{a} \in \overline{W} \end{cases}$$
$$= \begin{cases} \mathbf{x}+\mathbf{w}_1 & \text{if } \mathbf{x} \in \mathbf{a}+W \\ \mathbf{x}+\mathbf{w}_2 & \text{if } \mathbf{x} \in \mathbf{a}+\overline{W} \end{cases} = \begin{cases} m(\mathbf{x}) & \text{if } \mathbf{a} \in W, \\ \overline{m}(\mathbf{x}) & \text{if } \mathbf{a} \in \overline{W}. \end{cases} \qquad \square$$

Proposition 3. *We have*

$$m t_{\mathbf{a}} m = \overline{m} t_{\mathbf{a}} \overline{m} = \begin{cases} t_{\mathbf{a}} & \text{if } \mathbf{a} \in W, \\ t_{\mathbf{a}+\mathbf{w}_1+\mathbf{w}_2} & \text{if } \mathbf{a} \in \overline{W}. \end{cases}$$

Proof. If $\mathbf{a} \in W$ then Proposition 2 yields $t_{\mathbf{a}} m t_{\mathbf{a}} = m$, whence also $m t_{\mathbf{a}} m = t_{\mathbf{a}}$. If $\mathbf{a} \in \overline{W}$ then $t_{\mathbf{a}} m t_{\mathbf{a}} = \overline{m}$ by Proposition 2, whence $t_{\mathbf{a}} m t_{\mathbf{a}} m = \overline{m} m$ and $m t_{\mathbf{a}} m = t_{\mathbf{a}} \overline{m} m$; Proposition 1 yields $m t_{\mathbf{a}} m = t_{\mathbf{a}} t_{\mathbf{w}_1+\mathbf{w}_2} = t_{\mathbf{a}+\mathbf{w}_1+\mathbf{w}_2}$ and the assertion is proved. $\square$

Proposition 4. *If* $\mathbf{a} \in W$ *then we have* $m_{W,\mathbf{w}_1,\mathbf{w}_2} t_{\mathbf{a}} = m_{W,\mathbf{w}_1+\mathbf{a},\mathbf{w}_2+\mathbf{a}}$ *and* $\overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2} t_{\mathbf{a}} = \overline{m}_{W,\mathbf{w}_1+\mathbf{a},\mathbf{w}_2+\mathbf{a}}$.

Proof. By direct computation,

$$m t_{\mathbf{a}}(\mathbf{x}) = m(\mathbf{x}+\mathbf{a}) = \begin{cases} \mathbf{x}+\mathbf{w}_1+\mathbf{a} & \text{if } \mathbf{x} \in W, \\ \mathbf{x}+\mathbf{w}_2+\mathbf{a} & \text{if } \mathbf{x} \in \overline{W}, \end{cases}$$

$$\overline{m} t_{\mathbf{a}}(\mathbf{x}) = \overline{m}(\mathbf{x}+\mathbf{a}) = \begin{cases} \mathbf{x}+\mathbf{w}_1+\mathbf{a} & \text{if } \mathbf{x} \in \overline{W}, \\ \mathbf{x}+\mathbf{w}_2+\mathbf{a} & \text{if } \mathbf{x} \in W. \end{cases} \qquad \square$$

Proposition 5. *The relation* $\overline{m} m \overline{m} = m$ *holds.*

Proof. Using Proposition 1 we have

$$\overline{m} m \overline{m} = \overline{m} t_{\mathbf{w}_1+\mathbf{w}_2} = \overline{m}_{W,\mathbf{w}_1+\mathbf{w}_1+\mathbf{w}_2,\mathbf{w}_2+\mathbf{w}_1+\mathbf{w}_2} = \overline{m}_{W,\mathbf{w}_2,\mathbf{w}_1} = m_{W,\mathbf{w}_1,\mathbf{w}_2} = m$$

whence also $m \overline{m} m = \overline{m}$. $\square$

We observe that if $m_{W,\mathbf{w}_1,\mathbf{w}_2}$, $\overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2}$ are complementary mixed translations and g is an affine transformation of $AG(n,2)$ then we either have $gmg^{-1} = m_{W^g,\mathbf{w}_1^g,\mathbf{w}_2^g}$ and $g\overline{m}g^{-1} = \overline{m}_{W^g,\mathbf{w}_1^g,\mathbf{w}_2^g}$ or $gmg^{-1} = \overline{m}_{W^g,\mathbf{w}_1^g,\mathbf{w}_2^g}$ and $g\overline{m}g^{-1} = m_{W^g,\mathbf{w}_1^g,\mathbf{w}_2^g}$.

3. Primitive one-factorizations

Let d be a 2-primitive divisor of 2^n-1, $n \geqslant 4$. That is, a divisor of 2^n-1 such that d does not divide 2^m-1 for $m<n$. For $n \neq 6$ the existence of such a divisor is assured by Zsigmondy's Lemma, see, for instance, [7, Theorem 6.2]. We choose $V = GF(2^n)$ and define B to be the subgroup of order d of the multiplicative group $GF(2^n)\backslash\{0\}$. We define G to be the set of all mappings $g: V \to V$ of the form $g(\mathbf{x}) = b \cdot \mathbf{x} + \mathbf{c}$ for some element $b \in B$ and some vector $\mathbf{c} \in V$. When we regard V as an n-dimensional vector space over the field of two elements we have that G is a subgroup of $AGL(V) = AGL(n,2)$ acting primitively on V. In fact, since G is clearly transitive on V as it contains the translation group T, the primitivity of G can be proved by showing that the stabilizer $G_0 \cong B$ is a maximal subgroup of G. Namely, if we assume the existence of a proper subgroup H of G which contains G_0, then H consists of all transformations

$x \mapsto b \cdot \mathbf{x} + \mathbf{c}$, where $\mathbf{c}$ runs over a proper additive subgroup of $GF(2^n)$. Thus $|H| = |B| \cdot 2^m = d \cdot 2^m$ with $m < n$. The discussion in [6, II.8.7] shows that such a subgroup exists only when d is a divisor of $(2^m - 1)$, contradicting the choice of d.

The group $T = \{t_\mathbf{a} : \mathbf{a} \in V\}$ of all translations of V is an elementary abelian normal subgroup of G. Each translation $t_\mathbf{a} : \mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$, with $\mathbf{a} \neq 0$, is a fixed-point-free involution on V. The set $T \setminus \{\mathrm{id}_V\}$ of all non-trivial translations is a one-factorization of K_V, which is the affine line parallelism of $AG(n, 2)$, see [1, p. 10; 2, Section 2].

Since T is a normal subgroup of G, we have that this one-factorization is left invariant by G. It is even invariant under the larger group $AGL(n, 2)$ of all affine transformations. The G-orbit of any translation $t_\mathbf{a} \in T \setminus \{\mathrm{id}_V\}$, say $O_\mathbf{a}$, has length $|G_0| = d$. In fact we can decompose each transformation $g \in G \setminus T$ in the form $g = t_\mathbf{c} g_b$ with a suitable translation $t_\mathbf{c}$ and a mapping $g_b : V \to V$, $\mathbf{v} \mapsto b\mathbf{v}$. We have $(t_\mathbf{c} g_b)^{-1}(\mathbf{v}) = b^{-1}\mathbf{v} + b^{-1}\mathbf{c}$ and we can now verify the relation $g t_\mathbf{a} g^{-1} = (t_\mathbf{c} g_b) t_\mathbf{a} (t_\mathbf{c} g_b)^{-1} = t_{b\mathbf{a}}$. If $b \neq 1$ we have $b\mathbf{a} \neq \mathbf{a}$, which means that for every $t_\mathbf{a} \in T \setminus \{\mathrm{id}_V\}$ the centralizer in G of $t_\mathbf{a}$ is the translation group T, whence $|O_\mathbf{a}| = |G : T| = d$.

We shall now show that a suitable blend of translations and mixed translations yields a G-invariant one-factorization $\mathscr{F}'$ of K_V which is different from the affine line parallelism of $AG(n, 2)$.

As remarked in Section 2 a mixed translation $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$ depends on the choice of the hyperplane W and of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$ within W. Here, we consider an arbitrary hyperplane W of V and we choose the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ in such a way that the translations $t_{\mathbf{w}_1}$ and $t_{\mathbf{w}_2}$ do not lie in the same G-orbit. That amounts to saying that the B-multiples of $\mathbf{w}_1$ and the B-multiples of $\mathbf{w}_2$ are pairwise distinct within the field $GF(2^n)$. We will denote by m the mixed translation arising from this choice of $W, \mathbf{w}_1, \mathbf{w}_2$. Let O_m denote the G-orbit of the mixed translation m. We note that the transformations in O_m are the mixed translations $m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ and $\overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ with $b \in B$. In fact, writing $g \in G$ in the form $g = t_\mathbf{c} g_b$ as above, we obtain

$$gmg^{-1}(\mathbf{x}) = \begin{cases} g(g^{-1}(\mathbf{x}) + \mathbf{w}_1) & \text{if } g^{-1}(\mathbf{x}) \in W \\ g(g^{-1}(\mathbf{x}) + \mathbf{w}_2) & \text{if } g^{-1}(\mathbf{x}) \in \overline{W} \end{cases}$$
$$= \begin{cases} \mathbf{x} + b\mathbf{w}_1 & \text{if } \mathbf{x} \in bW + \mathbf{c}, \\ \mathbf{x} + b\mathbf{w}_2 & \text{if } \mathbf{x} \in b\overline{W} + \mathbf{c}. \end{cases}$$

We observe that if $\mathbf{c} \in bW$, then we have $bW + \mathbf{c} = bW$ and consequently $gmg^{-1} = m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$; if, instead, $\mathbf{c} \notin bW$, we obtain $gmg^{-1} = \overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$.

Proposition 6. *We have* $|O_m| = 2d$.

Proof. Let L denote the subgroup of index $2d$ in G consisting of all translations $t_\mathbf{a} \in T$ with $\mathbf{a} \in W$. We prove that L is precisely the centralizer in G of the mixed translation m.

Clearly, each translation in L fixes m by Proposition 2. For each translation $g \in T \setminus L$ we have $gmg^{-1} = \overline{m}$ by Proposition 2. For each transformation $g \in G \setminus T$, say $g = t_\mathbf{c} g_b$, we have that gmg^{-1} is either the mixed translation $m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ or its complementary mixed translation $\overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$, as remarked above. The definition of a mixed translation shows that either relation $m_{bW,b\mathbf{w}_1,b\mathbf{w}_2} = m$ and $\overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2} = m$ implies $b(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{w}_1 + \mathbf{w}_2$. As $\mathbf{w}_1 + \mathbf{w}_2$ is a non-zero vector, that forces $b = 1$, which is not the case by our choice of $g \in G \setminus T$. $\square$

Proposition 7. *If $g \in G$ is such that there exists $\mathbf{x} \in V$ with $gmg^{-1}(\mathbf{x}) = m(\mathbf{x})$, then we have $gmg^{-1} = m$.*

Proof. We write g in the form $g(\mathbf{x}) = b\mathbf{x} + \mathbf{c}$, for every $\mathbf{x} \in V$, with $b \in B$ and $\mathbf{c} \in V$. As already remarked the transformation gmg^{-1} is either the mixed translation $m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ or its complementary mixed translation $\overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$. Observe that bW is a hyperplane of V which may or may not coincide with W, in the latter case $bW \cap W$ is an $(n-2)$-dimensional vector subspace over $GF(2)$ of V.

We consider the case $bW = W$. We assume first $gmg^{-1} = m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$. The hypothesis in our statement implies $\mathbf{x} + b\mathbf{w}_1 = \mathbf{x} + \mathbf{w}_1$ if $\mathbf{x} \in W$ or $\mathbf{x} + b\mathbf{w}_2 = \mathbf{x} + \mathbf{w}_2$ if $\mathbf{x} \notin W$, respectively. That forces $b = 1$ in either case, whence $gmg^{-1} = m$. We assume next $gmg^{-1} = \overline{m}_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$. The hypothesis in our statement implies $\mathbf{x} + b\mathbf{w}_2 = \mathbf{x} + \mathbf{w}_1$ if $\mathbf{x} \in W$ or $\mathbf{x} + b\mathbf{w}_1 = \mathbf{x} + \mathbf{w}_2$ if $\mathbf{x} \notin W$, yielding in turn $t_{\mathbf{w}_1} \in O_{\mathbf{w}_2}$ or $t_{\mathbf{w}_2} \in O_{\mathbf{w}_1}$, respectively. In either case that contradicts the assumption that $t_{\mathbf{w}_1}$ and $t_{\mathbf{w}_2}$ do not belong to the same G-orbit.

We consider the case $bW \neq W$. We assume first $gmg^{-1} = m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$. The hypothesis in our statement implies $\mathbf{x} + b\mathbf{w}_1 = \mathbf{x} + \mathbf{w}_1$ if $\mathbf{x} \in bW \cap W$, $\mathbf{x} + b\mathbf{w}_1 = \mathbf{x} + \mathbf{w}_2$ if $\mathbf{x} \in bW \cap \overline{W}$, $\mathbf{x} + b\mathbf{w}_2 = \mathbf{x} + \mathbf{w}_1$ if $\mathbf{x} \in b\overline{W} \cap W$ or $\mathbf{x} + b\mathbf{w}_2 = \mathbf{x} + \mathbf{w}_2$ if $\mathbf{x} \in b\overline{W} \cap \overline{W}$, respectively. If $\mathbf{x} \in bW \cap W$ or $\mathbf{x} \in b\overline{W} \cap \overline{W}$ we obtain $b = 1$, contradicting $bW \neq W$. If $\mathbf{x} \in bW \cap \overline{W}$ we obtain $t_{\mathbf{w}_2} \in O_{\mathbf{w}_1}$, while if $\mathbf{x} \in b\overline{W} \cap W$ we obtain $t_{\mathbf{w}_1} \in O_{\mathbf{w}_2}$, respectively. In either case that contradicts the assumption that $t_{\mathbf{w}_1}$ and $t_{\mathbf{w}_2}$ do not belong to the same G-orbit. $\square$

Proposition 8. *Let g and h be elements in G with $gmg^{-1} \neq hmh^{-1}$. Then the one-factors gmg^{-1} and hmh^{-1} have no edge in common.*

Proof. Assume $gmg^{-1}(\mathbf{x}) = hmh^{-1}(\mathbf{x})$, for an arbitrary $\mathbf{x} \in V$. Setting $h^{-1}(\mathbf{x}) = \mathbf{y}$ we obtain $(h^{-1}g)m(h^{-1}g)^{-1}(\mathbf{y}) = m(\mathbf{y})$. Then by Proposition 7 we have $(h^{-1}g)m(h^{-1}g)^{-1} = m$, that is, $gmg^{-1} = hmh^{-1}$, which is a contradiction. $\square$

Proposition 9. *The set*

$$\mathscr{F}' = (T \backslash (\{\mathrm{id}_V\} \cup O_{\mathbf{w}_1} \cup O_{\mathbf{w}_2})) \cup O_m$$

is a one-factorization of K_V which is G-invariant.

Proof. Clearly, $\mathscr{F}'$ consists of permutations which are either translations or mixed translations on V, hence fixed-point-free involutions. The $2d$ translations in $O_{\mathbf{w}_1} \cup O_{\mathbf{w}_2}$ have been replaced by the $2d$ mixed translations in O_m and so the cardinality of $\mathscr{F}'$ is $2^n - 1$ as required.

In order to prove that $\mathscr{F}'$ is a one-factorization of K_V it suffices to show that, for every $f_1, f_2 \in \mathscr{F}'$, the relation $f_1(\mathbf{x}) = f_2(\mathbf{x})$ for some $\mathbf{x} \in V$ implies $f_1 = f_2$. That is clear if both f_1 and f_2 are translations. Proposition 7 shows that the same thing happens if both f_1 and f_2 belong to O_m. We consider the case in which $f_1 = t_{\mathbf{a}}$ is a translation and $f_2 = m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ is a mixed translation in O_m. There exists an element $\mathbf{x}$ in V for which either $\mathbf{x} + b\mathbf{w}_1 = \mathbf{x} + \mathbf{a}$ or $\mathbf{x} + b\mathbf{w}_2 = \mathbf{x} + \mathbf{a}$ holds; the former relation yields $b\mathbf{w}_1 = \mathbf{a}$, the latter one yields $b\mathbf{w}_2 = \mathbf{a}$, either one is a contradiction as $t_{\mathbf{a}}$ belongs to neither orbit $O_{\mathbf{w}_1}, O_{\mathbf{w}_2}$.

In order to prove that $\mathscr{F}'$ is G-invariant we must make sure that $\mathscr{F}'$ contains the G-orbit of any one of its elements, say f. If $f \in O_m$, then its G-orbit is O_m, which is contained in $\mathscr{F}'$ by the very definition. If $f = t_{\mathbf{a}}$ is a translation in $T \backslash (\{\mathrm{id}_V\} \cup O_{\mathbf{w}_1} \cup O_{\mathbf{w}_2})$, then, since T is G-invariant, the whole G-orbit $O_{\mathbf{a}}$ is also a subset of $T \backslash (\{\mathrm{id}_V\} \cup O_{\mathbf{w}_1} \cup O_{\mathbf{w}_2})$. $\square$

The one-factorization $\mathscr{F}'$ of Proposition 9 is primitive, because so is the group G. Since the number of vertices is a power of 2, we see that this one-factorization $\mathscr{F}'$ cannot be doubly transitive, otherwise it should arise from the affine line parallelism of $AG(n, 2)$ by the classification of Cameron and Korchmáros [2, Theorem 3]. But that is not the case since some of the one-factors in $\mathscr{F}'$ contain lines with different directions.

4. Live one-factorizations

It is the purpose of this section to give an example of a live one-factorization other than the affine line parallelism of $AG(n, 2)$.

Proposition 10. *Let $\mathbf{w}_1, \mathbf{w}_2$ be linearly independent vectors in W. Then $\mathscr{F} = \{m_{W,\mathbf{w}_1,\mathbf{w}_2}, \overline{m}_{W,\mathbf{w}_1,\mathbf{w}_2}\} \cup \{t_{\mathbf{a}} : \mathbf{a} \in V \backslash \{\mathbf{w}_1, \mathbf{w}_2\}, \mathbf{a} \neq \mathbf{0}\}$ is a live one-factorization of K_V.*

Proof. In order to prove that $\mathscr{F}$ is a one-factorization we have to show that the pairs $\{\mathbf{x}, f(\mathbf{x})\}$ as $\mathbf{x}$ varies in V and f varies in $\mathscr{F}$, yield all lines of $AG(V) = AG(n, 2)$. The pairs $\{\mathbf{x}, t_{\mathbf{a}}(\mathbf{x})\}$ as $\mathbf{a}$ varies in $V \backslash \{\mathbf{w}_1, \mathbf{w}_2\}$, $\mathbf{a} \neq \mathbf{0}$, yield all lines whose direction is different from $\mathbf{w}_1, \mathbf{w}_2$. The pairs $\{\mathbf{x}, m(\mathbf{x})\}$ as $\mathbf{x}$ varies in W and the pairs $\{\mathbf{x}, \overline{m}(\mathbf{x})\}$ as $\mathbf{x}$ varies in $\overline{W}$ yield all lines whose direction is $\mathbf{w}_1$. The pairs $\{\mathbf{x}, m(\mathbf{x})\}$ as $\mathbf{x}$ varies in $\overline{W}$ and the pairs $\{\mathbf{x}, \overline{m}(\mathbf{x})\}$ as $\mathbf{x}$ varies in W yield all lines whose direction is $\mathbf{w}_2$.

The one-factorization $\mathscr{F}$ is live if the relation $f\mathscr{F}f = \mathscr{F}$ holds for every $f \in \mathscr{F}$.

(i) If $f = m$ we have

$$fmf = mmm = m,$$

$$f\overline{m}f = m\overline{m}m = \overline{m}$$

by Proposition 5 and

$$ft_{\mathbf{a}}f = mt_{\mathbf{a}}m = \begin{cases} t_{\mathbf{a}} & \text{if } \mathbf{a} \in W, \\ t_{\mathbf{a}+\mathbf{w}_1+\mathbf{w}_2} & \text{if } \mathbf{a} \in \overline{W} \end{cases}$$

by Proposition 3. Now since $\mathbf{w}_1 + \mathbf{w}_2 \in W$ we have that the mapping $\overline{W} \to \overline{W}$, $\mathbf{a} \mapsto \mathbf{a} + \mathbf{w}_1 + \mathbf{w}_2$ is bijective.

(ii) If $f = \overline{m}$ the proof is analogous to (i).

(iii) If $f = t_{\mathbf{z}}$ with $\mathbf{z} \in V \backslash \{\mathbf{w}_1, \mathbf{w}_2\}$, $\mathbf{z} \neq \mathbf{0}$, we have

$$fmf = t_{\mathbf{z}}mt_{\mathbf{z}} = \begin{cases} m & \text{if } \mathbf{z} \in W, \\ \overline{m} & \text{if } \mathbf{z} \in \overline{W}, \end{cases}$$

$$f\overline{m}f = t_{\mathbf{z}}\overline{m}t_{\mathbf{z}} = \begin{cases} \overline{m} & \text{if } \mathbf{z} \in W, \\ m & \text{if } \mathbf{z} \in \overline{W}, \end{cases}$$

$$ft_{\mathbf{a}}f = t_{\mathbf{z}}t_{\mathbf{a}}t_{\mathbf{z}} = t_{\mathbf{a}} \quad \text{for every vector } \mathbf{a} \in V \backslash \{\mathbf{w}_1, \mathbf{w}_2\}.$$

That concludes the proof. □

5. Uniform and sequentially uniform one-factorizations

A one-factorization $\{F_1, \ldots, F_{2n-1}\}$ of K_{2n} is *uniform* if the graphs with edge sets $F_i \cup F_j$ are all isomorphic for all $1 \leqslant i < j \leqslant 2n-1$, see [1, p. 72]. A one-factorization $\{F_1, \ldots, F_{2n-1}\}$ of K_{2n} is *sequentially uniform* if the one-factors can be arranged in a cyclic order $(F_1, \ldots, F_{2n-1})$ in such a way that the graphs with edge sets $F_i \cup F_{i+1}$ are isomorphic for all $1 \leqslant i \leqslant 2n-1$. We say the multiset $T = (k_1, \ldots, k_r)$ is the type of a sequentially uniform one-factorization if the graph with edge set $F_i \cup F_{i+1}$ is the disjoint union of cycles of lengths $k_1, \ldots, k_r$, where $k_1 + \cdots + k_r = 2n$. In case $r = 1$ then we say that the one-factorization is sequentially *perfect*. These terminologies have been introduced in [4, Section 1, p. 2].

We know from [1, Theorem 4.5(ii)] that the affine line parallelism of $AG(n, 2)$ always yields a uniform one-factorization. The unique one-factorization of K_6 is also uniform, the union of two one-factors always yields a hamiltonian cycle here; hence we have a so-called *perfect* one-factorization in this case.

The unique doubly transitive one-factorization of K_{12}, see [2, Theorem 3], is uniform, the union of any two one-factors decomposes into two cycles of length 6 each. We exhibit this one-factorization here:

{1, 2}	{3, 12}	{4, 5}	{6, 7}	{8, 9}	{10, 11},
{1, 3}	{2, 8}	{4, 10}	{5, 6}	{7, 11}	{9, 12},
{1, 4}	{2, 3}	{5, 12}	{6, 11}	{7, 9}	{8, 10},
{1, 5}	{2, 10}	{3, 9}	{4, 11}	{6, 12}	{7, 8},
{1, 6}	{2, 5}	{3, 8}	{4, 7}	{9, 11}	{10, 12},
{1, 7}	{2, 12}	{3, 6}	{4, 8}	{5, 11}	{9, 10},
{1, 8}	{2, 7}	{3, 11}	{4, 12}	{5, 10}	{6, 9},
{1, 9}	{2, 4}	{3, 7}	{5, 8}	{6, 10}	{11, 12},
{1, 10}	{2, 9}	{3, 5}	{4, 6}	{7, 12}	{8, 11},
{1, 11}	{2, 6}	{3, 4}	{5, 9}	{7, 10}	{8, 12},
{1, 12}	{2, 11}	{3, 10}	{4, 9}	{5, 7}	{6, 8}.

The unique doubly transitive one-factorization of K_{28}, see [2, Theorem 3], is NOT uniform, the union of two one-factors either yields a hamiltonian cycle or splits into two cycles of length 14 each. It can be seen, however, that the order given by the following list of one-factors yields a sequentially perfect one-factorization:

{1, 8} {2, 4} {3, 9} {5, 21} {6, 7} {10, 11} {12, 19} {13, 20} {14, 18} {15, 26} {16, 17} {22, 27} {23, 28} {24, 25},
{1, 4} {2, 18} {3, 22} {5, 19} {6, 10} {7, 27} {8, 25} {9, 15} {11, 13} {12, 17} {14, 20} {16, 26} {21, 28} {23, 24},
{1, 3} {2, 15} {4, 8} {5, 6} {7, 14} {9, 13} {10, 16} {11, 12} {17, 27} {18, 19} {20, 24} {21, 22} {23, 25} {26, 28},
{1, 5} {2, 11} {3, 19} {4, 10} {6, 26} {7, 15} {8, 21} {9, 12} {13, 27} {14, 24} {16, 23} {17, 28} {18, 25} {20, 22},
{1, 6} {2, 23} {3, 4} {5, 12} {7, 18} {8, 27} {9, 14} {10, 15} {11, 21} {13, 25} {16, 24} {17, 26} {19, 20} {22, 28},
{1, 9} {2, 6} {3, 28} {4, 20} {5, 11} {7, 22} {8, 17} {10, 14} {12, 27} {13, 18} {15, 19} {16, 25} {21, 24} {23, 26},
{1, 2} {3, 24} {4, 6} {5, 22} {7, 17} {8, 19} {9, 10} {11, 23} {12, 13} {14, 16} {15, 21} {18, 20} {25, 26} {27, 28},
{1, 10} {2, 27} {3, 16} {4, 7} {5, 9} {6, 13} {8, 14} {11, 24} {12, 20} {15, 17} {18, 28} {19, 26} {21, 23} {22, 25},
{1, 7} {2, 21} {3, 5} {4, 26} {6, 24} {8, 13} {9, 22} {10, 12} {11, 28} {14, 27} {15, 16} {17, 18} {19, 25} {20, 23},
{1, 14} {2, 8} {3, 21} {4, 24} {5, 17} {6, 12} {7, 23} {9, 28} {10, 13} {11, 15} {16, 18} {19, 22} {20, 25} {26, 27},
{1, 16} {2, 17} {3, 10} {4, 14} {5, 13} {6, 9} {7, 8} {11, 20} {12, 24} {15, 27} {18, 26} {19, 28} {21, 25} {22, 23},
{1, 19} {2, 5} {3, 13} {4, 25} {6, 20} {7, 11} {8, 16} {9, 26} {10, 22} {12, 23} {14, 17} {15, 28} {18, 21} {24, 27},
{1, 15} {2, 9} {3, 26} {4, 22} {5, 25} {6, 18} {7, 13} {8, 28} {10, 23} {11, 14} {12, 16} {17, 20} {19, 24} {21, 27},
{1, 12} {2, 3} {4, 17} {5, 7} {6, 25} {8, 10} {9, 24} {11, 26} {13, 16} {14, 28} {15, 23} {18, 22} {19, 27} {20, 21},
{1, 18} {2, 13} {3, 15} {4, 9} {5, 26} {6, 22} {7, 25} {8, 20} {10, 24} {11, 17} {12, 28} {14, 19} {16, 21} {23, 27},
{1, 11} {2, 28} {3, 27} {4, 19} {5, 20} {6, 16} {7, 10} {8, 9} {12, 18} {13, 23} {14, 15} {17, 25} {21, 26} {22, 24},
{1, 17} {2, 22} {3, 18} {4, 12} {5, 27} {6, 8} {7, 19} {9, 23} {10, 25} {11, 16} {13, 26} {14, 21} {15, 24} {20, 28},
{1, 13} {2, 7} {3, 14} {4, 5} {6, 23} {8, 26} {9, 11} {10, 17} {12, 22} {15, 18} {16, 20} {19, 21} {24, 28} {25, 27},
{1, 20} {2, 12} {3, 25} {4, 13} {5, 23} {6, 19} {7, 21} {8, 24} {9, 17} {10, 28} {11, 18} {14, 26} {15, 22} {16, 27},
{1, 23} {2, 14} {3, 8} {4, 21} {5, 24} {6, 17} {7, 12} {9, 16} {10, 26} {11, 22} {13, 19} {15, 20} {18, 27} {25, 28},
{1, 21} {2, 10} {3, 12} {4, 11} {5, 8} {6, 15} {7, 26} {9, 19} {13, 22} {14, 25} {16, 28} {17, 23} {18, 24} {20, 27},
{1, 25} {2, 26} {3, 7} {4, 27} {5, 16} {6, 28} {8, 11} {9, 20} {10, 18} {12, 21} {13, 15} {14, 22} {17, 24} {19, 23},
{1, 24} {2, 16} {3, 23} {4, 18} {5, 15} {6, 14} {7, 20} {8, 22} {9, 25} {10, 19} {11, 27} {12, 26} {13, 28} {17, 21},
{1, 27} {2, 19} {3, 20} {4, 16} {5, 10} {6, 11} {7, 28} {8, 15} {9, 21} {12, 25} {13, 14} {17, 22} {18, 23} {24, 26},
{1, 22} {2, 24} {3, 11} {4, 23} {5, 28} {6, 21} {7, 9} {8, 18} {10, 27} {12, 14} {13, 17} {15, 25} {16, 19} {20, 26},
{1, 28} {2, 20} {3, 6} {4, 15} {5, 18} {7, 16} {8, 12} {9, 27} {10, 21} {11, 25} {13, 24} {14, 23} {17, 19} {22, 26},
{1, 26} {2, 25} {3, 17} {4, 28} {5, 14} {6, 27} {7, 24} {8, 23} {9, 18} {10, 20} {11, 19} {12, 15} {13, 21} {16, 22}.

We want to show that the live one-factorizations of the previous section are sequentially uniform but not uniform. It is easy to see that the union of two one-factors given by distinct translations, say $t_{\mathbf{a}}$ and $t_{\mathbf{b}}$, is the disjoint union of 2^{n-2} cycles of length 4. In fact a cycle in $t_{\mathbf{a}} \cup t_{\mathbf{b}}$ is obtained by translating each line which is parallel to $\mathbf{b}$ in the direction of $\mathbf{a}$. Now each translation $t_{\mathbf{a}}$ maps each class of parallel lines to itself. That means, if we start from the endpoints of a line in the direction $\mathbf{b}$ and follow the direction $\mathbf{a}$, we reach two points belonging to another line with direction $\mathbf{b}$ and obtain thus a cycle of length 4. That shows, in particular, that in the one-factorization arising from the affine line parallelism of $AG(n, 2)$ the union of any two one-factors decomposes into cycles all having length 4, see again [1, Theorem 4.5(ii)].

If, instead, we join a one-factor $t_{\mathbf{a}}$ and a mixed translation $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$, we have to translate the lines in W and $\overline{W}$ under $t_{\mathbf{a}}$. If $\mathbf{a} \in W$ then a line in W (in $\overline{W}$) is mapped to another line of W (of $\overline{W}$) and so we can repeat the argument above; that is, we obtain cycles of length 4. If $\mathbf{a} \notin W$ we obtain cycles of length 8. In fact let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a line of W with direction $\mathbf{w}_1$. If from the endpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ we follow the direction $\mathbf{a}$, we reach two points of $\overline{W}$, say $\mathbf{x}_3$ and $\mathbf{x}_4$. Since the lines we have chosen in $\overline{W}$ all have direction $\mathbf{w}_2$, from $\mathbf{x}_3$ and $\mathbf{x}_4$ we reach two more points, say $\mathbf{x}_5$ and $\mathbf{x}_6$. Observe that the lines $\{\mathbf{x}_3, \mathbf{x}_4\}$ and $\{\mathbf{x}_5, \mathbf{x}_6\}$ have direction $\mathbf{w}_1$, since each translation maps a parallel class to itself. Travelling in the direction $\mathbf{a}$ from $\mathbf{x}_5$ and $\mathbf{x}_6$ we reach two points, say $\mathbf{x}_7$ and $\mathbf{x}_8$, on a line with direction$\mathbf{w}_1$. We have thus completed the cycle of length 8.

Using the remarks above we can say that if we join a mixed translation $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$ and its complementary mixed translation $\overline{m} = m_{W,\mathbf{w}_1,\mathbf{w}_2}$, then all of the resulting cycles have length 4, since $\mathbf{w}_1$ and $\mathbf{w}_2$ lie in W.

The next case we consider is $m \cup m'$, where $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$ and $m' = m_{U,\mathbf{w}_3,\mathbf{w}_4}$. If $W = U$, then $\mathbf{w}_3$ and $\mathbf{w}_4$ both belong to W, so that all cycles in $m \cup m'$ have length 4. If $W \neq U$, then $|W \cap U| = 2^{n-2}$ and we must consider the possibilities $\mathbf{w}_3, \mathbf{w}_4 \in W$, or $\mathbf{w}_3 \in W, \mathbf{w}_4 \notin W$, or $\mathbf{w}_3, \mathbf{w}_4 \notin W$. If $\mathbf{w}_3$ and $\mathbf{w}_4$ are both in W, then all cycles in $m \cup m'$ have length 4. If $\mathbf{w}_3, \mathbf{w}_4 \notin W$ then all cycles in $m \cup m'$ have length 8. If $\mathbf{w}_3 \in W$ but $\mathbf{w}_4 \notin W$, then all cycles in $W \cap U$ and $\overline{W} \cap U$ have length 4, whereas all cycles in $\overline{U}$ have length 8.

The previous remarks allow us to say that the live one-factorization $\mathscr{F}$ we construct in Proposition 10 is not uniform, because of the presence of cycles of lengths 4 and 8. Nevertheless it is sequentially uniform of type $(4, \ldots, 4)$ whenever $n \geqslant 4$. In fact, let $m = m_{W,\mathbf{w}_1,\mathbf{w}_2}$ and $\overline{m} = m_{W,\mathbf{w}_1,\mathbf{w}_2}$ be the mixed translations contained in $\mathscr{F}$. We know that all cycles in $m \cup \overline{m}$ have length 4. For every $n \geqslant 4$ there exist at least two one-factors, say $t_{\mathbf{a}_1}, t_{\mathbf{a}_2} \in \mathscr{F}$, such that $\mathbf{a}_1, \mathbf{a}_2 \in W$, so that all cycles in $\overline{m} \cup t_{\mathbf{a}_1}$ and in $m \cup t_{\mathbf{a}_2}$ have length 4. Let $t_{\mathbf{b}_i}$, $i = 1, 2, \ldots, 2^n - 5$ denote the one-factors of $\mathscr{F}$ arising from the translations different from $t_{\mathbf{a}_1}$ and $t_{\mathbf{a}_2}$. Then the sequence $(m, \overline{m}, t_{\mathbf{a}_1}, t_{\mathbf{b}_1}, \ldots, t_{\mathbf{b}_{2^n-5}}, t_{\mathbf{a}_2})$ is of type $(4, \ldots, 4)$, since all cycles in the union of two one-factors arising from two distinct translations have length 4.

Next we consider the primitive one-factorization $\mathscr{F}'$ we construct in Proposition 9. We can say that it is certainly not uniform. As a matter of fact the union of two one-factors arising from two complementary mixed translations decomposes into cycles all having length 4. On the other hand, we can certainly find two one-factors whose union contains cycles of length 8. That occurs with a mixed translation with, say, hyperplane W, together with another transformation moving some of the points into a direction not lying in W: that can be achieved by suitably choosing as the second transformation either a mixed translation which is not the complementary mixed translation of the previous one, or one of the surviving translations.

Is $\mathscr{F}'$ always sequentially uniform? We do not address this question here. Nevertheless we are able to find an example of a primitive sequentially uniform one-factorization of K_{16} which is of type $(8, 8)$:

{1, 2}	{3, 6}	{4, 5}	{7, 8}	{9, 10}	{11, 14}	{12, 13}	{15, 16},
{1, 3}	{2, 14}	{4, 16}	{5, 7}	{6, 10}	{8, 12}	{9, 11}	{13, 15},
{1, 6}	{2, 5}	{3, 8}	{4, 7}	{9, 14}	{10, 13}	{11, 16}	{12, 15},
{1, 8}	{2, 7}	{3, 4}	{5, 6}	{9, 16}	{10, 15}	{11, 12}	{13, 14},
{1, 4}	{2, 12}	{3, 9}	{5, 15}	{6, 7}	{8, 14}	{10, 11}	{13, 16},
{1, 5}	{2, 6}	{3, 7}	{4, 8}	{9, 13}	{10, 14}	{11, 15}	{12, 16},
{1, 11}	{2, 3}	{4, 10}	{5, 8}	{6, 16}	{7, 13}	{9, 12}	{14, 15},
{1, 9}	{2, 10}	{3, 16}	{4, 15}	{5, 13}	{6, 14}	{7, 12}	{8, 11},
{1, 7}	{2, 8}	{3, 5}	{4, 6}	{9, 15}	{10, 16}	{11, 13}	{12, 14},
{1, 12}	{2, 11}	{3, 14}	{4, 13}	{5, 16}	{6, 15}	{7, 10}	{8, 9},
{1, 10}	{2, 9}	{3, 12}	{4, 11}	{5, 14}	{6, 13}	{7, 16}	{8, 15},
{1, 13}	{2, 4}	{3, 15}	{5, 9}	{6, 8}	{7, 11}	{10, 12}	{14, 16},
{1, 16}	{2, 15}	{3, 10}	{4, 9}	{5, 12}	{6, 11}	{7, 14}	{8, 13},
{1, 14}	{2, 13}	{3, 11}	{4, 12}	{5, 10}	{6, 9}	{7, 15}	{8, 16},
{1, 15}	{2, 16}	{3, 13}	{4, 14}	{5, 11}	{6, 12}	{7, 9}	{8, 10}.

If, instead, in every sequence of one-factors of $\mathscr{F}'$ we always have at least two elements of O_m which are adjacent, then if $\mathscr{F}'$ is sequentially uniform, it might be of type $(4, \ldots, 4)$ or $(8, \ldots, 8)$. Observe that if $m = m_{bW,b\mathbf{w}_1,b\mathbf{w}_2}$ and $m' = m_{cW,c\mathbf{w}_1,c\mathbf{w}_2}$ are elements of O_m, with $bW \neq cW$ and $c\mathbf{w}_1 \in bW$ but $c\mathbf{w}_2 \notin bW$, which are adjacent in every sequence of one-factors of $\mathscr{F}'$, then $\mathscr{F}'$ is not sequentially uniform. In fact in $m \cup m'$ we find cycles of length 4 and cycles of length 8, but in a sequence of one-factors of $\mathscr{F}'$ we have at least a one-factor $t_{\mathbf{a}}$ which is adjacent to a mixed translation m'', so that in their union all the cycles have length 4 or 8; that is, $m \cup m'$ is not isomorphic to $t_{\mathbf{a}} \cup m''$.

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